

Well-posedness and decay for the dissipative system modeling electro-hydrodynamics in negative Besov spaces

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Abstract

In [10] (Y. Guo, Y. Wang, Decay of dissipative equations and negative Sobolev spaces, Commun. Partial Differ. Equ. 37 (2012) 2165–2208), Y. Guo and Y. Wang developed a general new energy method for proving the optimal time decay rates of the solutions to dissipative equations. In this paper, we generalize this method in the framework of homogeneous Besov spaces. Moreover, we apply this method to a model arising from electro-hydrodynamics, which is a strongly coupled system of the Navier-Stokes equations and the Poisson-Nernst-Planck equations through charge transport and external forcing terms. We show that the negative Besov norms are preserved along time evolution, and obtain the optimal time decay rates of the higher-order spatial derivatives of solutions by the Fourier splitting approach and the interpolation techniques.

Keywords: Navier-Stokes equations; Poisson-Nernst-Planck equations; electro-hydrodynamics; well-posedness; decay; Besov space

2010 AMS Subject Classification: 35B40, 35K15, 35K55, 35Q35, 76A05

1 Introduction

In [10], Y. Guo and Y. Wang developed a new energy approach to establish the optimal time decay rates of the solutions to the Cauchy problem of the heat equation:

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}^3, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \quad (1.1)$$

They proved the following result:

Theorem 1.1 *If $u_0 \in H^N(\mathbb{R}^3) \cap \dot{H}^{-s}(\mathbb{R}^3)$ with $N \geq 0$ be an integer and $s \geq 0$ be a real number, then for any real number $\ell \in [-s, N]$, there exists a constant C_0 such that*

$$\|\nabla^\ell u(t)\|_{L^2} \leq C_0(1+t)^{-\frac{\ell+s}{2}}. \quad (1.2)$$

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Here $H^s(\mathbb{R}^3)$ and $\dot{H}^s(\mathbb{R}^3)$ denote the nonhomogeneous Sobolev space and the homogeneous Sobolev space, respectively.

In this paper, we generalize this new energy approach in the framework of Besov spaces. In order to illustrate this approach, we revisit the heat equation (1.1).

Theorem 1.2 *Let $N \geq 0$ be an integer and $s \geq 0$ be a real number, $1 \leq p < \infty$. If $u_0 \in \dot{B}_{p,1}^N(\mathbb{R}^3) \cap \dot{B}_{p,1}^{-s}(\mathbb{R}^3)$, then for any real number $\ell \in [-s, N]$, there exists a constant C_0 such that*

$$\|u(t)\|_{\dot{B}_{p,1}^\ell} \leq C_0(1+t)^{-\frac{\ell+s}{2}}. \quad (1.3)$$

Proof. Let $\ell \in [-s, N]$. Applying the dyadic operator Δ_j to the heat equation (1.1), we see that

$$\partial_t \Delta_j u - \Delta \Delta_j u = 0,$$

which taking the standard L^2 inner product with $|\Delta_j u|^{p-2} \Delta_j u$ leads to

$$\frac{1}{p} \frac{d}{dt} \|\Delta_j u\|_{L^p}^p - \int_{\mathbb{R}^3} \Delta \Delta_j u |\Delta_j u|^{p-2} \Delta_j u dx = 0.$$

Thanks to [5], there exists a constant κ such that

$$- \int_{\mathbb{R}^3} \Delta \Delta_j u |\Delta_j u|^{p-2} \Delta_j u dx \geq \kappa 2^{2j} \|\Delta_j u\|_{L^p}^p.$$

Thus, we obtain

$$\frac{d}{dt} \|\Delta_j u\|_{L^p} + \kappa 2^{2j} \|\Delta_j u\|_{L^p} \leq 0.$$

Multiplying the above inequality by $2^{j\ell}$, then taking l^1 norm to the resultant yields that

$$\frac{d}{dt} \|u\|_{\dot{B}_{p,1}^\ell} + \kappa \|u\|_{\dot{B}_{p,1}^{\ell+2}} \leq 0. \quad (1.4)$$

Integrating the above in time, we obtain

$$\|u\|_{\dot{B}_{p,1}^\ell} \leq \|u_0\|_{\dot{B}_{p,1}^\ell}. \quad (1.5)$$

This implies that inequality (1.3) holds in particular with $\ell = -s$. Now for $-s < \ell \leq N$, we use the interpolation relation, see Lemma 5.2 below, to get

$$\|u\|_{\dot{B}_{p,1}^\ell} \leq \|u\|_{\dot{B}_{p,1}^{-s}}^{\frac{2}{\ell+s+2}} \|u\|_{\dot{B}_{p,1}^{\ell+2}}^{\frac{\ell+s}{\ell+s+2}},$$

which combining (1.5) implies that

$$\|u\|_{\dot{B}_{p,1}^{\ell+2}} \geq \|u_0\|_{\dot{B}_{p,1}^{-s}}^{-\frac{2}{\ell+s}} \|u\|_{\dot{B}_{p,1}^\ell}^{1+\frac{2}{\ell+s}}. \quad (1.6)$$

Plugging (1.6) into (1.4), we conclude that there exists a constant C_0 such that

$$\frac{d}{dt} \|u\|_{\dot{B}_{p,1}^\ell} + C_0 \|u\|_{\dot{B}_{p,1}^\ell}^{1+\frac{2}{\ell+s}} \leq 0.$$

Solving this inequality implies that

$$\|u\|_{\dot{B}_{p,1}^\ell} \leq (\|u_0\|_{\dot{B}_{p,1}^\ell}^{-\frac{2}{\ell+s}} + \frac{2C_0 t}{\ell+s})^{-\frac{\ell+s}{2}} \leq C_0(1+t)^{-\frac{\ell+s}{2}}.$$

We complete the proof of Theorem 1.2. \square

Organization of the paper In Section 2, we make some preliminary preparations. In Section 3, we state our main results. Section 4 is devoted to giving the proofs of Theorems 3.1 and 3.2. In the final Appendix, we first collect some analytic tools used in this paper, then give a sketched proof of the global existence of solutions with small initial data in Theorem 3.1.

2 Preliminaries

2.1 Notations

In this paper, we shall use the following notations.

- For two constants A and B , the notation $A \lesssim B$ means that there is a uniform constant C (always independent of x, t), which may vary from line to line, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.
- For a quasi-Banach space X and for any $0 < T \leq \infty$, we use standard notation $L^p(0, T; X)$ or $L_T^p(X)$ for the quasi-Banach space of Bochner measurable functions f from $(0, T)$ to X endowed with the norm

$$\|f\|_{L_T^p(X)} := \begin{cases} (\int_0^T \|f(\cdot, t)\|_X^p dt)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \sup_{0 \leq t \leq T} \|f(\cdot, t)\|_X & \text{for } p = \infty. \end{cases}$$

In particular, if $T = \infty$, we use $\|f\|_{L_t^p(X)}$ instead of $\|f\|_{L_\infty^p(X)}$.

- We shall denote by $(f|g)$ the $L^2(\mathbb{R}^3)$ inner product of two functions f and g .
- $(d_j)_{j \in \mathbb{Z}}$ will be a generic element of $l^1(\mathbb{Z})$ so that $d_j \geq 0$ and $\sum_{j \in \mathbb{Z}} d_j = 1$.
- We say that a vector $u = (u^1, u^2, u^3)$ belongs to a function space X if $u^j \in X$ holds for every $j = 1, 2, 3$ and we put $\|u\|_X := \max_{1 \leq j \leq 3} \|u^j\|_X$.
- Given two quasi-Banach spaces X and Y , the product of these two spaces $X \times Y$ will be equipped with the usual norm $\|(u, v)\|_{X \times Y} := \|u\|_X + \|v\|_Y$.

2.2 Littlewood-Paley theory and Besov spaces

Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function, and $\mathcal{S}'(\mathbb{R}^3)$ of temperate distributions be the dual set of $\mathcal{S}(\mathbb{R}^3)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^3)$ be a smooth radial function valued in $[0, 1]$ such that φ is supported in the shell $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Then for any $f \in \mathcal{S}'(\mathbb{R}^3)$, we define for all $j \in \mathbb{Z}$,

$$\Delta_j f := \varphi(2^{-j}D)f \quad \text{and} \quad S_j f := \sum_{k \leq j-1} \Delta_k f. \quad (2.1)$$

By telescoping the series, we have the following homogeneous Littlewood-Paley decomposition:

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{for } f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3),$$

where $\mathcal{P}(\mathbb{R}^3)$ is the set of polynomials (see [1]). We remark here that the Littlewood-Paley decomposition satisfies the property of almost orthogonality, that is to say, for any $f, g \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$, the following properties hold:

$$\Delta_i \Delta_j f \equiv 0 \quad \text{if } |i - j| \geq 2 \quad \text{and} \quad \Delta_i (S_{j-1} f \Delta_j g) \equiv 0 \quad \text{if } |i - j| \geq 5. \quad (2.2)$$

Using the above decomposition, the stationary/time dependent homogeneous Besov spaces can be defined as follows:

Definition 2.1 Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|f\|_{\dot{B}_{p,r}^s} := \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j f\|_{L^p}^r \right)^{\frac{1}{r}} & \text{for } 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{L^p} & \text{for } r = \infty. \end{cases}$$

Then the homogeneous Besov space $\dot{B}_{p,r}^s(\mathbb{R}^3)$ is defined by

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define

$$\dot{B}_{p,r}^s(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) : \|f\|_{\dot{B}_{p,r}^s} < \infty \right\}.$$

- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \leq s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}_{p,r}^s(\mathbb{R}^3)$ is defined as the subset of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta f \in \mathcal{S}'(\mathbb{R}^3)$ whenever $|\beta| = k$.

Definition 2.2 ([4]) For $0 < T \leq \infty$, $s \leq \frac{3}{p}$ (resp. $s \in \mathbb{R}$), $1 \leq p, r, \rho \leq \infty$. We define the mixed time-space $\mathcal{L}^\rho(0, T; \dot{B}_{p,r}^s(\mathbb{R}^3))$ as the completion of $\mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}^3))$ by the norm

$$\|f\|_{\mathcal{L}_T^\rho(\dot{B}_{p,r}^s)} := \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \left(\int_0^T \|\Delta_j f(\cdot, t)\|_{L^p}^\rho dt \right)^{\frac{r}{\rho}} \right)^{\frac{1}{r}} < \infty$$

with the usual change if $\rho = \infty$ or $r = \infty$. For simplicity, we use $\|f\|_{\mathcal{L}_t^\rho(\dot{B}_{p,r}^s)}$ instead of $\|f\|_{\mathcal{L}_\infty^\rho(\dot{B}_{p,r}^s)}$.

The following properties of Besov spaces are well-known:

- (1) If $s < \frac{3}{p}$ or $s = \frac{3}{p}$ and $r = 1$, then $(\dot{B}_{p,r}^s(\mathbb{R}^3), \|\cdot\|_{\dot{B}_{p,r}^s})$ is a Banach space which is continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$.

- (2) In the case that $p = r = 2$, we get the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3) \cong \dot{B}_{2,2}^s(\mathbb{R}^3)$, which is endowed with the equivalent norm $\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}$ with $\Lambda = \sqrt{-\Delta}$.

(3) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$, and $u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$. Then $u \in \dot{B}_{p,r}^s(\mathbb{R}^3)$ if and only if there exists $\{d_{j,r}\}_{j \in \mathbb{Z}}$ such that $d_{j,r} \geq 0$, $\|d_{j,r}\|_{l^r} = 1$ and

$$\|\Delta_j u\|_{L^p} \lesssim d_{j,r} 2^{-js} \|u\|_{\dot{B}_{p,r}^s} \quad \text{for all } j \in \mathbb{Z}.$$

(4) According to the Minkowski inequality, it is readily to see that

$$\begin{cases} \|f\|_{\mathcal{L}_T^\rho(\dot{B}_{p,r}^s)} \leq \|f\|_{L_T^\rho(\dot{B}_{p,r}^s)} & \text{if } \rho \leq r, \\ \|f\|_{L_T^\rho(\dot{B}_{p,r}^s)} \leq \|f\|_{\mathcal{L}_T^\rho(\dot{B}_{p,r}^s)} & \text{if } r \leq \rho. \end{cases} \quad (2.3)$$

Finally we recall the following Bony's paradifferential decomposition (see [3]). The paraproduct between f and g is defined by

$$T_f g := \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g.$$

Thus we have the formal decomposition

$$fg = T_f g + T_g f + R(f, g),$$

where

$$R(f, g) := \sum_{j \in \mathbb{Z}} \Delta_j f \widetilde{\Delta_j g} \quad \text{and} \quad \widetilde{\Delta_j} := \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

3 Main results

We are concerned with the following system of dissipative nonlinear equations governing hydrodynamic transport of binary diffuse charge densities. The 3-D Cauchy problem reads as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = \varepsilon \Delta \phi \nabla \phi, & x \in \mathbb{R}^3, \ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^3, \ t > 0, \\ \partial_t v + u \cdot \nabla v = \nabla \cdot (D_1 \nabla v - \nu_1 v \nabla \phi), & x \in \mathbb{R}^3, \ t > 0, \\ \partial_t w + u \cdot \nabla w = \nabla \cdot (D_2 \nabla w + \nu_2 w \nabla \phi), & x \in \mathbb{R}^3, \ t > 0, \\ \varepsilon \Delta \phi = v - w, & x \in \mathbb{R}^3, \ t > 0 \end{cases} \quad (3.1)$$

with initial condition

$$(u, v, w)|_{t=0} = (u_0, v_0, w_0), \quad x \in \mathbb{R}^3. \quad (3.2)$$

Here u and Π denote the velocity field and the pressure of the fluid, respectively, ϕ is the electrostatic potential caused by the charged particles, v and w denote the charge densities of a negatively and positively charged species, respectively, hence the sign difference in front of the convective term in either equation. μ is the kinematic viscosity, and ε is the dielectric constant, known as the Debye length, related to vacuum permittivity and characteristic charge density. D_1 , D_2 , ν_1 , ν_2 are the diffusion and mobility coefficients of the charged particles¹. Since the concrete values of the

¹ $D_1 = \frac{kT_0\nu_1}{e}$, $D_2 = \frac{kT_0\nu_2}{e}$, where T_0 is the ambient temperature, k is the Boltzmann constant, and e is the charge mobility.

constants $\mu, \varepsilon, D_1, D_2, \nu_1$ and ν_2 play no role in our discussion, for simplicity, we shall assume them to be all equal to one throughout this paper.

We mention here that the right-hand side term in the momentum equations is the Lorentz force, which exhibits $\varepsilon \Delta \phi \nabla \phi = \varepsilon \nabla \cdot \sigma$, where the electric stress σ is a rank one tensor plus a pressure, for $i, j = 1, 2, 3$,

$$[\sigma]_{ij} = (\nabla \phi \otimes \nabla \phi - \frac{1}{2} |\nabla \phi|^2 I)_{ij} = \partial_{x_i} \phi \partial_{x_j} \phi - \frac{1}{2} |\nabla \phi|^2 \delta_{ij}. \quad (3.3)$$

Here I is 3×3 identity matrix, δ_{ij} is the Kronecker symbol, and \otimes denotes the tensor product. The electric stress σ stems from the balance of kinetic energy with electrostatic energy via the least action principle (cf. [20]).

The system (3.1)–(3.2) was introduced by Rubinstein [18], which is capable of describing electrochemical and fluid-mechanical transport throughout the cellular environment. At the present time, modeling of electro-diffusion in electrolytes is a problem of major scientific interest, it finds that such model has a wide applications in biology (ion channels), chemistry (electro-osmosis) and pharmacology (transdermal iontophoresis), we refer the readers to see [14]–[16] for the computational simulations, and [2], [8], [9], [22] for detailed applications of the system (3.1)–(3.2). The mathematical analysis of the system (3.1)–(3.2) was initiated by Jerome [11], where the author established a local existence–uniqueness theory of the system (3.1)–(3.2) based on the Kato’s semi-group framework. For more results concerning existence of (large) weak solutions, (small and local) mild solutions, convergence rate estimates to stationary solutions of time-dependent solutions and other related topics we refer the reader to see [7], [12], [13], [19], [21], [23], [24] and the reference therein.

The invariant space for solving the system (3.1)–(3.2) requires us to analyze the scaling invariance property of the system (3.1)–(3.2). Set

$$(u_\lambda, v_\lambda, w_\lambda, \Pi_\lambda, \phi_\lambda)(x, t) := (\lambda u, \lambda^2 v, \lambda^2 w, \lambda^2 \Pi, \phi)(\lambda x, \lambda^2 t).$$

Then if (u, v, w) solves (3.1) with initial data (u_0, v_0, w_0) (Π, ϕ can be determined by (u, v, w)), so does $(u_\lambda, v_\lambda, w_\lambda)$ with initial data $(u_{0\lambda}, v_{0\lambda}, w_{0\lambda})$ ($\Pi_\lambda, \phi_\lambda$ can be determined by $(u_\lambda, v_\lambda, w_\lambda)$), where $u_{0\lambda}(x) := \lambda u_0(\lambda x)$, $v_{0\lambda}(x) := \lambda^2 v_0(\lambda x)$, $w_{0\lambda}(x) := \lambda^2 w_0(\lambda x)$. In particular, the norm of $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, $(v_0, w_0) \in \dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3)$ ($1 \leq p, q \leq \infty$) are scaling invariant under the above change of scale.

Motivated by the optimal time decay rates of the solutions to the heat equation in the framework of Besov spaces, we aim at using this approach to the system (3.1)–(3.2). The main results are as follows:

Theorem 3.1 *Let p, q be two positive numbers such that $1 \leq p < \infty$, $1 \leq q < 6$, and*

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{3}, \quad \frac{1}{q} - \frac{1}{p} > -\min\{\frac{1}{3}, \frac{1}{2p}\}.$$

Suppose that $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $v_0, w_0 \in \dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3)$. Then there exists a positive constant η such that if

$$\|(u_0, v_0, w_0)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}} \times (\dot{B}_{q,1}^{-2+\frac{3}{q}})^2} \leq \eta,$$

then the system (3.1)–(3.2) admits a unique solution (u, v, w) satisfying

$$\begin{cases} u \in C([0, \infty), \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0, \infty; \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)), \\ v, w \in C([0, \infty), \dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0, \infty; \dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3)) \cap L^1(0, \infty; \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)). \end{cases}$$

If we assume further that $u_0 \in \dot{B}_{r,1}^{-s}(\mathbb{R}^3) \cap \dot{B}_{r,1}^N(\mathbb{R}^3)$, $v_0, w_0 \in \dot{B}_{r,1}^{-s-1}(\mathbb{R}^3) \cap \dot{B}_{r,1}^{N-1}(\mathbb{R}^3)$ for an integer N , a real number $s > 0$ and $1 < r < \infty$ such that

$$\frac{3}{p} - s > 3 \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad \frac{3}{q} - s > 3 \max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

then for any $\ell \in [-s, N]$, there exists a constant C_0 such that for all $t \geq 0$,

$$\|(u(t), v(t), w(t))\|_{\dot{B}_{r,1}^\ell \times (\dot{B}_{r,1}^{\ell-1})^2} \leq C_0. \quad (3.4)$$

Moreover, we have

$$\|(u(t), v(t), w(t))\|_{\dot{B}_{r,1}^\ell \times (\dot{B}_{r,1}^{\ell-1})^2} \leq C_0(1+t)^{-\left(\frac{\ell+s}{2}\right)}. \quad (3.5)$$

If we relax the high regularity condition imposed on the initial data in Theorem 3.1, then we can obtain the following decay result.

Theorem 3.2 *Under the assumptions of Theorem 3.1. Assume that (u, v, w) be a unique global solution corresponding to the initial data (u_0, v_0, w_0) . If we assume further that $u_0 \in \dot{B}_{r,1}^{-s}(\mathbb{R}^3)$, $v_0, w_0 \in \dot{B}_{r,1}^{-s-1}(\mathbb{R}^3)$ with $1 < r \leq \min\{p, q\}$, $s > \max\{0, 2 - \frac{3}{r}\}$, and*

$$\frac{3}{p} - s > 3 \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad \frac{3}{q} - s > 3 \max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

then for any $\ell \in [-s - 3(\frac{1}{r} - \frac{1}{p}), -1 + \frac{3}{p}]$, there exists a constant C_0 such that for all $t \geq 0$,

$$\|u(t)\|_{\dot{B}_{r,1}^\ell} \leq C_0(1+t)^{-\left(\frac{\ell+s}{2}\right) - \frac{3}{2}(\frac{1}{r} - \frac{1}{p})}; \quad (3.6)$$

for any $\ell \in [-s - 1 - 3(\frac{1}{r} - \frac{1}{q}), -2 + \frac{3}{q}]$, there exists a constant C_0 such that for all $t \geq 0$,

$$\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}} \leq C_0(1+t)^{-\left(\frac{\ell+s}{2}\right) - \frac{3}{2}(\frac{1}{r} - \frac{1}{q})}. \quad (3.7)$$

We emphasize here that in [24], the authors in this paper and Zhang established global well-posedness of the system (3.1)–(3.2) in the critical Besov spaces $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3) \times (\dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3))^2$ with $1 \leq p < \infty$ and $1 \leq q < 6$, $q \leq p$ and $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$. We relax the restrictive condition $q \leq p$ in Theorem 3.1. The main observation is that we can convert the estimation of $\Delta\phi\nabla\phi$ into the estimation of $v\nabla(-\Delta)^{-1}w + w\nabla(-\Delta)^{-1}v$ via the fifth equation of (3.1), which has a nice structure as follows: for $1 \leq m \leq 3$,

$$\begin{aligned} (v\nabla(-\Delta)^{-1}w + w\nabla(-\Delta)^{-1}v)_m &= (-\Delta)\left\{((- \Delta)^{-1}v)(\partial_m(-\Delta)^{-1}w)\right\} \\ &\quad + 2\nabla \cdot \left\{((- \Delta)^{-1}v)(\partial_m\nabla(-\Delta)^{-1}w)\right\} + \partial_m\left\{((- \Delta)^{-1}v)w\right\}. \end{aligned}$$

Thanks to this observation, the condition $q \leq p$ can be removed.

Another important feature in Theorems 3.1 and 3.2 is that the negative Besov norms of the solutions of the system (3.1)–(3.2) are preserved along the time evolution and enhance the time decay rates, see Proposition 4.6 below.

4 Proofs of Theorems 3.1 and 3.2

We aim at establishing two basic energy inequalities in the framework of Besov spaces, then prove Theorems 3.1 and 3.2 by using the approach illustrated in Theorem 1.2. For clarity of our statement, we leave the proof of global well-posedness of the system (3.1)–(3.2) with small initial data in Appendix.

4.1 Lower-order derivative estimates

We denote

$$\mathcal{E}(t) := \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}$$

and

$$Y(t) := \int_0^t (\|u(\tau)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(\tau), w(\tau))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}) d\tau.$$

Proposition 4.1 *Let p, q be two positive numbers such that $1 \leq p < \infty$, $1 \leq q < 6$, and*

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{3}, \quad \frac{1}{q} - \frac{1}{p} > -\min\{\frac{1}{3}, \frac{1}{2p}\}.$$

Assume that $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $v_0, w_0 \in \dot{B}_{q,1}^{-2+\frac{3}{q}}(\mathbb{R}^3)$. Let η be the number such that if

$$\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v_0, w_0)\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \leq \eta,$$

then the system (3.1)–(3.2) admits a unique solution (u, v, w) . In addition, there exist two constants κ and K such that the following inequality holds:

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{E}(t)) + \kappa e^{-KY(t)}(\|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}) \leq 0. \quad (4.1)$$

In order to prove Proposition 4.1, we define

$$\tilde{u} := e^{-KY(t)}u, \quad \tilde{v} := e^{-KY(t)}v, \quad \tilde{w} := e^{-KY(t)}w, \quad \tilde{\Pi} := e^{-KY(t)}\Pi, \quad \tilde{\phi} := e^{-KY(t)}\phi,$$

where K is a constant to be specified later. Then we see that $(\tilde{u}, \tilde{v}, \tilde{w})$ satisfies the following equations:

$$\begin{cases} \partial_t \tilde{u} + u \cdot \nabla \tilde{u} - \Delta \tilde{u} + \nabla \tilde{\Pi} = \Delta \tilde{\phi} \nabla \phi - KY'(t)\tilde{u}, \\ \nabla \cdot \tilde{u} = 0, \\ \partial_t \tilde{v} + u \cdot \nabla \tilde{v} = \nabla \cdot (\nabla \tilde{v} - \tilde{v} \nabla \phi) - KY'(t)\tilde{v}, \\ \partial_t \tilde{w} + u \cdot \nabla \tilde{w} = \nabla \cdot (\nabla \tilde{w} + \tilde{w} \nabla \phi) - KY'(t)\tilde{w}, \\ \Delta \tilde{\phi} = \tilde{v} - \tilde{w}. \end{cases} \quad (4.2)$$

Lemma 4.2 *Let $1 \leq p < \infty$. Then*

$$\|\Delta_j(u \cdot \nabla \tilde{u})\|_{L^p} \lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \quad (4.3)$$

Proof. Thanks to Bony's paraproduct decomposition, we have

$$u \cdot \nabla \tilde{u} = \nabla \cdot (u \otimes \tilde{u}) = \nabla \cdot (2T_{\tilde{u}}u + R(\tilde{u}, u)).$$

Moreover, applying Lemma 5.1 yields that

$$\begin{aligned} \|\Delta_j \nabla \cdot (T_{\tilde{u}}u)\|_{L^p} &\lesssim 2^j \sum_{|j-j'|\leq 4} \|S_{j'-1}\tilde{u}\|_{L^\infty} \|\Delta_{j'}u\|_{L^p} \\ &\lesssim 2^j \sum_{|j-j'|\leq 4} \sum_{k\leq j'-2} 2^k 2^{(-1+\frac{3}{p})k} \|\Delta_k \tilde{u}\|_{L^p} \|\Delta_{j'}u\|_{L^p} \\ &\lesssim 2^j \sum_{|j-j'|\leq 4} 2^{j'} \|\Delta_{j'}u\|_{L^p} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \end{aligned}$$

To estimate the remaining term $R(\tilde{u}, u)$, in the case $1 \leq p < 2$, there exists $2 < p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, thus we can deduce from Lemma 5.1 that

$$\begin{aligned} \|\Delta_j \nabla \cdot R(\tilde{u}, u)\|_{L^p} &\lesssim 2^{(4-\frac{3}{p})j} \sum_{j'\geq j-N_0} \|\Delta_{j'}\tilde{u}\|_{L^{p'}} \|\tilde{\Delta}_{j'}u\|_{L^p} \\ &\lesssim 2^{(4-\frac{3}{p})j} \sum_{j'\geq j-N_0} 2^{(-3+\frac{6}{p})j'} \|\Delta_{j'}\tilde{u}\|_{L^p} \|\tilde{\Delta}_{j'}u\|_{L^p} \\ &\lesssim 2^{(1-\frac{3}{p})j} \sum_{j'\geq j-N_0} 2^{-3(j'-j)} 2^{(-1+\frac{3}{p})j'} \|\Delta_{j'}\tilde{u}\|_{L^p} 2^{(1+\frac{3}{p})j'} \|\tilde{\Delta}_{j'}u\|_{L^p} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \end{aligned}$$

If $2 \leq p < \infty$, we estimate

$$\begin{aligned} \|\Delta_j \nabla \cdot R(\tilde{u}, u)\|_{L^p} &\lesssim 2^{(1+\frac{3}{p})j} \sum_{j'\geq j-N_0} \|\Delta_{j'}\tilde{u}\|_{L^p} \|\tilde{\Delta}_{j'}u\|_{L^p} \\ &\lesssim 2^{(1-\frac{3}{p})j} \sum_{j'\geq j-N_0} 2^{-\frac{6}{p}(j'-j)} 2^{(-1+\frac{3}{p})j'} \|\Delta_{j'}\tilde{u}\|_{L^p} 2^{(1+\frac{3}{p})j'} \|\tilde{\Delta}_{j'}u\|_{L^p} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \end{aligned}$$

This completes the proof of Lemma 4.2. \square

Lemma 4.3 *Let $1 \leq p, q < \infty$ and $\frac{1}{q} - \frac{1}{p} \geq -\min\{\frac{1}{3}, \frac{1}{2p}\}$. Then*

$$\|\Delta_j (\tilde{v} \nabla (-\Delta)^{-1} w + \tilde{w} \nabla (-\Delta)^{-1} v)\|_{L^p} \lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|(\tilde{v}, \tilde{w})\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.4)$$

Proof. The case $1 \leq q \leq p$ is simple. Indeed, based on the observation

$$v\nabla(-\Delta)^{-1}w + w\nabla(-\Delta)^{-1}v = \nabla \cdot (\nabla(-\Delta)^{-1}v\nabla(-\Delta)^{-1}w)$$

and the imbedding relation $\dot{B}_{q,1}^{-1+\frac{3}{q}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, we see that $\nabla(-\Delta)^{-1}v$ play the same role as u in Lemma 4.2. Therefore, we get the desired inequality (4.4). On the other hand, if $1 \leq p < q$, we resort to Bony's paraproduct decomposition to get

$$\tilde{v}\nabla(-\Delta)^{-1}w + \tilde{w}\nabla(-\Delta)^{-1}v := J_1 + J_2 + J_3, \quad (4.5)$$

where

$$\begin{aligned} J_1 &:= \sum_{j' \in \mathbb{Z}} S_{j'-1} \tilde{v} \nabla(-\Delta)^{-1} \Delta_{j'} w + S_{j'-1} \tilde{w} \nabla(-\Delta)^{-1} \Delta_{j'} v, \\ J_2 &:= \sum_{j' \in \mathbb{Z}} \Delta_{j'} v \nabla(-\Delta)^{-1} S_{j'-1} \tilde{w} + \Delta_{j'} w \nabla(-\Delta)^{-1} S_{j'-1} \tilde{v}, \\ J_3 &:= \sum_{j' \in \mathbb{Z}} \Delta_{j'} \tilde{v} \nabla(-\Delta)^{-1} \tilde{\Delta}_{j'} w + \Delta_{j'} \tilde{w} \nabla(-\Delta)^{-1} \tilde{\Delta}_{j'} v. \end{aligned}$$

For J_1 , it suffices to deal with the first term $\sum_{j' \in \mathbb{Z}} S_{j'-1} \tilde{v} \nabla(-\Delta)^{-1} \Delta_{j'} w$ because of the second one can be done analogously. Using the conditions $1 \leq p < q < \infty$ and $\frac{1}{q} - \frac{1}{p} \geq -\min\{\frac{1}{3}, \frac{1}{2p}\}$, we derive from Lemma 5.1 that

$$\begin{aligned} \|\Delta_j \sum_{j' \in \mathbb{Z}} S_{j'-1} \tilde{v} \nabla(-\Delta)^{-1} \Delta_{j'} w\|_{L^p} &\lesssim \sum_{|j-j'| \leq 4} \|S_{j'-1} \tilde{v}\|_{L^{\frac{pq}{q-p}}} \|\nabla(-\Delta)^{-1} \Delta_{j'} w\|_{L^q} \\ &\lesssim \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{3(\frac{1}{q} - \frac{q-p}{pq})k} \|\Delta_k \tilde{v}\|_{L^q} 2^{-j'} \|\Delta_{j'} w\|_{L^q} \\ &\lesssim \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{(2+\frac{3}{q}-\frac{3}{p})k} 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{v}\|_{L^q} 2^{-j'} \|\Delta_{j'} w\|_{L^q} \\ &\lesssim \sum_{|j-j'| \leq 4} 2^{(1+\frac{3}{q}-\frac{3}{p})j'} \|\Delta_{j'} w\|_{L^q} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\ &\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}, \end{aligned}$$

which directly leads to

$$\|\Delta_j J_1\|_{L^p} \lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|(\tilde{v}, \tilde{w})\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.6)$$

Similarly, for the first term of J_2 , we get

$$\begin{aligned} \|\Delta_j \sum_{j' \in \mathbb{Z}} \Delta_{j'} v \nabla(-\Delta)^{-1} S_{j'-1} \tilde{w}\|_{L^p} &\lesssim \sum_{|j-j'| \leq 4} \|\Delta_{j'} v\|_{L^q} \|\nabla(-\Delta)^{-1} S_{j'-1} \tilde{w}\|_{L^{\frac{pq}{q-p}}} \\ &\lesssim \sum_{|j-j'| \leq 4} \|\Delta_{j'} v\|_{L^q} \sum_{k \leq j'-2} 2^{[-1+3(\frac{1}{q}-\frac{q-p}{pq})]k} \|\Delta_k \tilde{w}\|_{L^q} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{|j-j'|\leq 4} \|\Delta_{j'} v\|_{L^q} \sum_{k\leq j'-2} 2^{(1+\frac{3}{q}-\frac{3}{p})k} 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{w}\|_{L^q} \\
&\lesssim \sum_{|j-j'|\leq 4} 2^{(1+\frac{3}{q}-\frac{3}{p})j'} \|\Delta_{j'} v\|_{L^q} \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{p})j} d_j \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}},
\end{aligned}$$

which yields that

$$\|\Delta_j J_2\|_{L^p} \lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|(\tilde{v}, \tilde{w})\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.7)$$

Finally we tackle with the most difficult term J_3 , the interesting observation is that we can split J_3 into the following three terms for $m = 1, 2, 3$:

$$J_3 := K_1 + K_2 + K_3, \quad (4.8)$$

where

$$\begin{aligned}
K_1 &:= \sum_{j' \in \mathbb{Z}} (-\Delta) \left\{ ((-\Delta)^{-1} \Delta_{j'} \tilde{v}) (\partial_m (-\Delta)^{-1} \tilde{\Delta}_{j'} w) \right\}, \\
K_2 &:= \sum_{j' \in \mathbb{Z}} 2 \nabla \cdot \left\{ ((-\Delta)^{-1} \Delta_{j'} \tilde{v}) (\partial_m \nabla (-\Delta)^{-1} \tilde{\Delta}_{j'} w) \right\}, \\
K_3 &:= \sum_{j' \in \mathbb{Z}} \partial_m \left\{ ((-\Delta)^{-1} \Delta_{j'} \tilde{v}) \tilde{\Delta}_{j'} w \right\}.
\end{aligned}$$

Since K_2 can be treated similarly to K_3 , we treat K_1 and K_3 only. It follows from Lemma 5.1 that

$$\begin{aligned}
\|\Delta_j K_1\|_{L^p} &\lesssim 2^{2j} \sum_{j' \geq j-N_0} \|(-\Delta)^{-1} \Delta_{j'} \tilde{v}\|_{L^{\frac{pq}{q-p}}} \|\partial_m (-\Delta)^{-1} \tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{2j} \sum_{j' \geq j-N_0} 2^{(-2+\frac{6}{q}-\frac{3}{p})j'} \|\Delta_{j'} \tilde{v}\|_{L^q} 2^{-j'} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{2j} \sum_{j' \geq j-N_0} 2^{-(1+\frac{3}{p})j'} 2^{(-2+\frac{3}{q})j'} \|\Delta_{j'} \tilde{v}\|_{L^q} 2^{\frac{3j'}{q}} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{p})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}},
\end{aligned}$$

$$\begin{aligned}
\|\Delta_j K_3\|_{L^p} &\lesssim 2^j \sum_{j' \geq j-N_0} \|(-\Delta)^{-1} \Delta_{j'} \tilde{v}\|_{L^{\frac{pq}{q-p}}} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^j \sum_{j' \geq j-N_0} 2^{-\frac{3}{p}j'} 2^{(-2+\frac{3}{q})j'} \|\Delta_{j'} \tilde{v}\|_{L^q} 2^{\frac{3j'}{q}} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{p})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}
\end{aligned}$$

$$\lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.$$

As a consequence, we deduce from (4.8) that

$$\|\Delta_j J_3\|_{L^p} \lesssim 2^{(1-\frac{3}{p})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.9)$$

Hence, plugging (4.6), (4.7) and (4.9) into (4.5), we obtain (4.4). The proof of Lemma 4.3 is complete. \square

Lemma 4.4 *Let $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$. Then we have*

$$\|\Delta_j(u \cdot \nabla \tilde{v})\|_{L^q} \lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) (\|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}). \quad (4.10)$$

Proof. Thanks to Bony's paraproduct decomposition, we have

$$u \cdot \nabla \tilde{v} = T_{\tilde{u}} \nabla v + T_{\nabla \tilde{v}} u + R(u, \nabla \tilde{v}).$$

Applying Lemma 5.1 gives us to

$$\begin{aligned} \|\Delta_j(T_{\tilde{u}} \nabla v)\|_{L^q} &\lesssim \sum_{|j'-j| \leq 4} 2^{j'} \|S_{j'-1} \tilde{u}\|_{L^\infty} \|\Delta_{j'} v\|_{L^q} \\ &\lesssim \sum_{|j'-j| \leq 4} 2^{j'} \sum_{k \leq j'-2} 2^{\frac{3k}{p}} \|\Delta_k \tilde{u}\|_{L^p} \|\Delta_{j'} v\|_{L^q} \\ &\lesssim \sum_{|j'-j| \leq 4} 2^{2j'} \|\Delta_{j'} v\|_{L^q} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \\ &\lesssim 2^{(2-\frac{3}{q})j} d_j \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \\ &\lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \end{aligned}$$

If $1 \leq q \leq p$, then there exists $1 < \lambda \leq \infty$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{\lambda}$, we calculate as

$$\begin{aligned} \|\Delta_j(T_{\nabla \tilde{v}} u)\|_{L^q} &\lesssim \sum_{|j'-j| \leq 4} \|\Delta_j S_{j'-1} \nabla \tilde{v}\|_{L^\lambda} \|\Delta_{j'} u\|_{L^p} \\ &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{(1+\frac{3}{p})k} \|\Delta_k \tilde{v}\|_{L^q} \|\Delta_{j'} u\|_{L^p} \\ &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{(3+\frac{3}{p}-\frac{3}{q})k} 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{v}\|_{L^q} \|\Delta_{j'} u\|_{L^p} \\ &\lesssim 2^{(2-\frac{3}{q})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\ &\lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \end{aligned}$$

If $1 \leq p < q$, we calculate as

$$\|\Delta_j(T_{\nabla \tilde{v}} u)\|_{L^q} \lesssim 2^{3(\frac{1}{p}-\frac{1}{q})j} \sum_{|j'-j| \leq 4} \|S_{j'-1} \nabla \tilde{v} \Delta_{j'} u\|_{L^p}$$

$$\begin{aligned}
&\lesssim 2^{3(\frac{1}{p}-\frac{1}{q})j} \sum_{|j'-j|\leq 4} \sum_{k\leq j'-2} 2^{(1+\frac{3}{q})k} \|\Delta_k \tilde{v}\|_{L^q} \|\Delta_{j'} u\|_{L^p} \\
&\lesssim 2^{3(\frac{1}{p}-\frac{1}{q})j} \sum_{|j'-j|\leq 4} \sum_{k\leq j'-2} 2^{3k} 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{v}\|_{L^q} \|\Delta_{j'} u\|_{L^p} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

To estimate the remaining term $R(u, \nabla \tilde{v})$, in the case that $\frac{1}{p} + \frac{1}{q} \leq 1$, the condition $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$ implies that

$$\begin{aligned}
\|\Delta_j R(u, \nabla \tilde{v})\|_{L^q} &\lesssim 2^{(1+\frac{3}{p})j} \sum_{j' \geq j-N_0} \|\Delta_{j'} u\|_{L^p} \|\tilde{\Delta}_{j'} \tilde{v}\|_{L^q} \\
&\lesssim 2^{(1+\frac{3}{p})j} \sum_{j' \geq j-N_0} 2^{(1-\frac{3}{p}-\frac{3}{q})j'} 2^{(1+\frac{3}{p})j'} \|\Delta_{j'} u\|_{L^p} 2^{(-2+\frac{3}{q})j'} \|\tilde{\Delta}_{j'} \tilde{v}\|_{L^q} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

In the case that $\frac{1}{p} + \frac{1}{q} > 1$, we find $1 < q' \leq \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\begin{aligned}
\|\Delta_j R(u, \nabla \tilde{v})\|_{L^q} &\lesssim 2^{j+3(1-\frac{1}{q})j} \sum_{j' \geq j-N_0} \|\Delta_{j'} u \tilde{\Delta}_{j'} \tilde{v}\|_{L^1} \\
&\lesssim 2^{(4-\frac{3}{q})j} \sum_{j' \geq j-N_0} \|\Delta_{j'} u\|_{L^{q'}} \|\tilde{\Delta}_{j'} \tilde{v}\|_{L^q} \\
&\lesssim 2^{(4-\frac{3}{q})j} \sum_{j' \geq j-N_0} 2^{-2j'} 2^{(1+\frac{3}{p})j'} \|\Delta_{j'} u\|_{L^p} 2^{(-2+\frac{3}{q})j'} \|\tilde{\Delta}_{j'} \tilde{v}\|_{L^q} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

We finish the proof of Lemma 4.4. \square

Lemma 4.5 *Let $1 \leq q < 6$. Then we have*

$$\|\Delta_j (\tilde{v} \nabla (-\Delta)^{-1} w)\|_{L^q} \lesssim 2^{(1-\frac{3}{q})j} d_j Y'(t) \|(\tilde{v}, \tilde{w})\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.11)$$

Proof. Thanks to Bony's paraproduct decomposition, we obtain

$$\tilde{v} \nabla (-\Delta)^{-1} w = T_{\tilde{v}} \nabla (-\Delta)^{-1} w + T_{\nabla (-\Delta)^{-1} \tilde{w}} \tilde{v} + R(\tilde{v}, \nabla (-\Delta)^{-1} w).$$

Applying Lemmas 5.1 and 5.2 yields that

$$\|\Delta_j (T_{\tilde{v}} \nabla (-\Delta)^{-1} w)\|_{L^q} \lesssim \sum_{|j'-j|\leq 4} \|S_{j'-1} \tilde{v}\|_{L^\infty} \|\Delta_{j'} \nabla (-\Delta)^{-1} w\|_{L^q}$$

$$\begin{aligned}
&\lesssim \sum_{|j'-j|\leq 4} 2^{-j'} \sum_{k\leq j'-2} 2^{\frac{3k}{q}} \|\Delta_k \tilde{w}\|_{L^q} \|\Delta_{j'} w\|_{L^q} \\
&\lesssim \sum_{|j'-j|\leq 4} 2^{(-1-\frac{3}{q})j'} \sum_{k\leq j'-2} 2^{2k} 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{w}\|_{L^q} 2^{\frac{3j'}{q}} \|\Delta_{j'} w\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j Y'(t) \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}},
\end{aligned}$$

$$\begin{aligned}
\|\Delta_j (T_{\nabla(-\Delta)^{-1} \tilde{w}} v)\|_{L^q} &\lesssim \sum_{|j'-j|\leq 4} \|S_{j'-1} \nabla(-\Delta)^{-1} \tilde{w}\|_{L^\infty} \|\Delta_{j'} v\|_{L^q} \\
&\lesssim \sum_{|j'-j|\leq 4} \sum_{k\leq j'-2} 2^{(-1+\frac{3}{q})k} \|\Delta_k \tilde{w}\|_{L^q} \|\Delta_{j'} v\|_{L^q} \\
&\lesssim \sum_{|j'-j|\leq 4} 2^{-\frac{3j'}{q}} \sum_{k\leq j'-2} 2^k 2^{(-2+\frac{3}{q})k} \|\Delta_k \tilde{w}\|_{L^q} 2^{\frac{3j'}{q}} \|\Delta_{j'} v\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j Y'(t) \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

Finally, in the case $1 \leq q < 2$, there exists $2 < q' \leq \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, thus using Lemma 5.1 yields that

$$\begin{aligned}
\|\Delta_j R(\tilde{v}, \nabla(-\Delta)^{-1} w)\|_{L^q} &\lesssim 2^{3(1-\frac{1}{q})j} \sum_{j' \geq j-N_0} \|\Delta_{j'} \tilde{v} \tilde{\Delta}_{j'} \nabla(-\Delta)^{-1} w\|_{L^1} \\
&\lesssim 2^{3(1-\frac{1}{q})j} \sum_{j' \geq j-N_0} \|\Delta_{j'} \tilde{v}\|_{L^{q'}} \|\tilde{\Delta}_{j'} \nabla(-\Delta)^{-1} w\|_{L^q} \\
&\lesssim 2^{3(1-\frac{1}{q})j} \sum_{j' \geq j-N_0} 2^{-2j'} 2^{(-2+\frac{3}{q})j'} \|\Delta_{j'} \tilde{v}\|_{L^q} 2^{\frac{3j'}{q}} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

In the case $2 \leq q < 6$, we get by using Lemma 5.1 again that

$$\begin{aligned}
\|\Delta_j R(\tilde{v}, \nabla(-\Delta)^{-1} w)\|_{L^q} &\lesssim 2^{\frac{3j}{q}} \sum_{j' \geq j-N_0} \|\Delta_{j'} \tilde{v}\|_{L^q} \|\tilde{\Delta}_{j'} \nabla(-\Delta)^{-1} w\|_{L^q} \\
&\lesssim 2^{\frac{3j}{q}} \sum_{j' \geq j-N_0} 2^{(1-\frac{6}{q})j'} 2^{(-2+\frac{3}{q})j'} \|\Delta_{j'} \tilde{v}\|_{L^q} 2^{\frac{3j'}{q}} \|\tilde{\Delta}_{j'} w\|_{L^q} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \\
&\lesssim 2^{(1-\frac{3}{q})j} d_j Y'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}.
\end{aligned}$$

We conclude that the proof of Lemma 4.5 is complete. \square

The estimates of u Applying the dyadic operator Δ_j to the first equation of (4.2), then multiplying $|\Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u}$ and integrating over \mathbb{R}^3 (when $p \in (1, 2)$, we need to make some modification as that in [5]), we see that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\Delta_j \tilde{u}\|_{L^p}^p - \int_{\mathbb{R}^3} \Delta \Delta_j \tilde{u} |\Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u} dx &= -(\Delta_j(u \cdot \nabla \tilde{u}) | \Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u}) \\ &+ (\Delta_j(\Delta \tilde{\phi} \nabla \phi) | \Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u}) - KY'(t) \|\Delta_j \tilde{u}\|_{L^p}^p \\ &\leq \|\Delta_j(u \cdot \nabla \tilde{u})\|_{L^p} \|\Delta_j \tilde{u}\|_{L^p}^{p-1} + \|\Delta_j(\Delta \tilde{\phi} \nabla \phi)\|_{L^p} \|\Delta_j \tilde{u}\|_{L^p}^{p-1} - KY'(t) \|\Delta_j \tilde{u}\|_{L^p}^p, \end{aligned} \quad (4.12)$$

where we have used the fact

$$\int_{\mathbb{R}^3} \nabla \Delta_j \tilde{\Pi} |\Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u} dx = 0,$$

which follows from the incompressibility condition $\nabla \cdot \tilde{u} = 0$. Thanks to [5, 17], there exists a positive constant κ so that

$$- \int_{\mathbb{R}^3} \Delta \Delta_j \tilde{u} \cdot |\Delta_j \tilde{u}|^{p-2} \Delta_j \tilde{u} dx \geq \kappa 2^{2j} \|\Delta_j \tilde{u}\|_{L^p}^p.$$

Therefore, we infer from (4.12) that

$$\frac{d}{dt} \|\Delta_j \tilde{u}\|_{L^p} + \kappa 2^{2j} \|\Delta_j \tilde{u}\|_{L^p} \lesssim \|\Delta_j(u \cdot \nabla \tilde{u})\|_{L^p} + \|\Delta_j(\Delta \tilde{\phi} \nabla \phi)\|_{L^p} - KY'(t) \|\Delta_j \tilde{u}\|_{L^p}.$$

Note that by the Poisson equation, i.e., the fifth equation of the system (3.1), we have

$$\Delta \tilde{\phi} \nabla \phi = -(\tilde{v} - \tilde{w}) \nabla (-\Delta)^{-1} (v - w).$$

Applying Lemmas 4.2 and 4.3 to get that

$$\frac{d}{dt} \|\Delta_j \tilde{u}\|_{L^p} + \kappa 2^{2j} \|\Delta_j \tilde{u}\|_{L^p} \leq C 2^{(1-\frac{3}{p})j} d_j Y'(t) e^{-KY(t)} \mathcal{E}(t) - KY'(t) \|\Delta_j \tilde{u}\|_{L^p},$$

which leads directly to

$$\frac{d}{dt} \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \kappa \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \leq CY'(t) e^{-KY(t)} \mathcal{E}(t) - KY'(t) \|\tilde{u}\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}. \quad (4.13)$$

The estimates of v and w We only show the desired estimates for \tilde{v} due to \tilde{w} can be done analogously as \tilde{v} . Applying the dyadic operator Δ_j to the third equation of (4.2), then multiplying $|\Delta_j \tilde{v}|^{q-2} \Delta_j \tilde{v}$ and integrating over \mathbb{R}^3 (when $q \in (1, 2)$, we need to make some modification as that in [5]), we see that

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \|\Delta_j \tilde{v}\|_{L^q}^q - \int_{\mathbb{R}^3} \Delta \Delta_j \tilde{v} |\Delta_j \tilde{v}|^{q-2} \Delta_j \tilde{v} dx &= -(\Delta_j(u \cdot \nabla \tilde{v}) | \Delta_j \tilde{v}|^{q-2} \Delta_j \tilde{v}) \\ &- (\Delta_j \nabla \cdot (\tilde{v} \nabla \phi) | \Delta_j \tilde{v}|^{q-2} \Delta_j \tilde{v}) - KY'(t) \|\Delta_j \tilde{v}\|_{L^q}^q \\ &\leq \|\Delta_j(u \cdot \nabla \tilde{v})\|_{L^q} \|\Delta_j \tilde{v}\|_{L^q}^{q-1} + \|\Delta_j \nabla \cdot (\tilde{v} \nabla \phi)\|_{L^q} \|\Delta_j \tilde{v}\|_{L^q}^{q-1} - KY'(t) \|\Delta_j \tilde{v}\|_{L^q}^q. \end{aligned} \quad (4.14)$$

Thanks to [5, 17], there exists a positive constant κ so that

$$- \int_{\mathbb{R}^3} \Delta \Delta_j \tilde{v} \cdot |\Delta_j \tilde{v}|^{q-2} \Delta_j \tilde{v} dx \geq \kappa 2^{2j} \|\Delta_j \tilde{v}\|_{L^q}^q.$$

Back to (4.14), we obtain that

$$\frac{d}{dt} \|\Delta_j \tilde{v}\|_{L^q} + \kappa 2^{2j} \|\Delta_j \tilde{v}\|_{L^q} \lesssim \|\Delta_j (u \cdot \nabla \tilde{v})\|_{L^q} + \|\Delta_j \nabla \cdot (\tilde{v} \nabla \phi)\|_{L^q} - KY'(t) \|\Delta_j \tilde{v}\|_{L^q}.$$

Lemmas 4.4 and 4.5 gives us to

$$\frac{d}{dt} \|\Delta_j \tilde{v}\|_{L^q} + \kappa 2^{2j} \|\Delta_j \tilde{v}\|_{L^q} \lesssim 2^{(2-\frac{3}{q})j} d_j Y'(t) e^{-KY(t)} \mathcal{E}(t) - KY'(t) \|\Delta_j \tilde{v}\|_{L^q},$$

which implies directly that

$$\frac{d}{dt} \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} + \kappa \|\tilde{v}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \leq CY'(t) e^{-KY(t)} \mathcal{E}(t) - KY'(t) \|\tilde{v}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.15)$$

Similarly, for w , we have

$$\frac{d}{dt} \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} + \kappa \|\tilde{w}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \leq CY'(t) e^{-KY(t)} \mathcal{E}(t) - KY'(t) \|\tilde{w}\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}. \quad (4.16)$$

Proof of Proposition 4.1 It is clear that from (4.13), (4.15)–(4.16), there exists a constant C such that

$$\frac{d}{dt} (e^{-KY(t)} \mathcal{E}(t)) + \kappa (\|\tilde{u}(t)\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(\tilde{v}(t), \tilde{w}(t))\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) \leq (C - K) Y'(t) e^{-KY(t)} \mathcal{E}(t).$$

By choosing K sufficiently large such that $K > C$, we see that

$$\frac{d}{dt} (e^{-KY(t)} \mathcal{E}(t)) + \kappa (\|\tilde{u}(t)\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(\tilde{v}(t), \tilde{w}(t))\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) \leq 0.$$

We finish the proof of Proposition 4.1.

4.2 Higher-order derivatives estimates

Next we derive the higher-order spatial derivatives of the solutions to the system (3.1)–(3.2). Let ℓ be a real number and $1 < r < \infty$. Define

$$\mathcal{F}(t) := \|u(t)\|_{\dot{B}_{r,1}^\ell} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}}.$$

We obtain the following result.

Proposition 4.6 *Under the assumptions of Proposition 4.1, if we further assume that $u_0 \in \dot{B}_{r,1}^\ell(\mathbb{R}^3)$, $v_0, w_0 \in \dot{B}_{r,1}^{\ell-1}(\mathbb{R}^3)$ with $1 < r < \infty$, and*

$$\frac{3}{p} + \ell > 3 \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad \frac{3}{q} + \ell > 3 \max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

then there exist two positive constants κ and K such that for all $t \geq 0$, the unique solution (u, v, w) of the system (3.1)–(3.2) satisfies

$$\frac{d}{dt} (e^{-KY(t)} \mathcal{F}(t)) + \frac{\kappa}{2} e^{-KY(t)} (\|u(t)\|_{\dot{B}_{r,1}^{\ell+2}} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell+1}}) \leq 0. \quad (4.17)$$

Proof. Applying the operator $\Delta_j \Lambda^\ell$ to the first equation of (3.1), and $\Delta_j \Lambda^{\ell-1}$ to the third equation of (3.1), then taking L^2 inner product with $|\Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u$ to the first resultant, and $|\Delta_j \Lambda^{\ell-1} v|^{r-2} \Delta_j \Lambda^{\ell-1} v$ to the second resultant (while for $1 < r < 2$, we need to make some modification as that in [5]), we obtain that

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|\Delta_j \Lambda^\ell u\|_{L^r}^r - (\Delta \Delta_j \Lambda^\ell u | \Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u) &= -(\Delta_j \Lambda^\ell (u \cdot \nabla u) | \Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u) \\ &\quad + (\Delta_j \Lambda^\ell (\Delta \phi \nabla \phi) | \Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u) \\ &\leq (\|\Delta_j \Lambda^\ell (u \cdot \nabla u)\|_{L^r} + \|\Delta_j \Lambda^\ell (\Delta \phi \nabla \phi)\|_{L^r}) \|\Delta_j \Lambda^\ell u\|_{L^r}^{r-1}, \end{aligned}$$

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|\Delta_j \Lambda^{\ell-1} v\|_{L^r}^r - (\Delta \Delta_j \Lambda^{\ell-1} v | \Delta_j \Lambda^{\ell-1} v|^{r-2} \Delta_j \Lambda^{\ell-1} v) &= -(\Delta_j \Lambda^{\ell-1} (u \cdot \nabla v) | \Delta_j \Lambda^{\ell-1} v|^{r-2} \Delta_j \Lambda^{\ell-1} v) \\ &\quad - (\Delta_j \Lambda^{\ell-1} \nabla \cdot (v \nabla \phi) | \Delta_j \Lambda^{\ell-1} v|^{r-2} \Delta_j \Lambda^{\ell-1} v) \\ &\leq (\|\Delta_j \Lambda^{\ell-1} (u \cdot \nabla v)\|_{L^r} + \|\Delta_j \Lambda^{\ell-1} \nabla \cdot (v \nabla \phi)\|_{L^r}) \|\Delta_j \Lambda^{\ell-1} v\|_{L^r}^{r-1}, \end{aligned}$$

where we have used the fact

$$\int_{\mathbb{R}^3} \nabla \Delta_j \Lambda^\ell \Pi | \Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u dx = 0,$$

which follows from the incompressibility condition $\nabla \cdot u = 0$. Thanks again to [5, 17], there exists a positive constant κ so that

$$\begin{aligned} - \int_{\mathbb{R}^3} \Delta \Delta_j \Lambda^\ell u \cdot | \Delta_j \Lambda^\ell u|^{r-2} \Delta_j \Lambda^\ell u dx &\geq \kappa 2^{2j} \|\Delta_j \Lambda^\ell u\|_{L^r}^r, \\ - \int_{\mathbb{R}^3} \Delta \Delta_j \Lambda^{\ell-1} v \cdot | \Delta_j \Lambda^{\ell-1} v|^{r-2} \Delta_j \Lambda^{\ell-1} v dx &\geq \kappa 2^{2j} \|\Delta_j \Lambda^{\ell-1} v\|_{L^r}^r. \end{aligned}$$

It follows that

$$\frac{d}{dt} \|\Delta_j \Lambda^\ell u\|_{L^r} + \kappa 2^{2j} \|\Delta_j \Lambda^\ell u\|_{L^r} \lesssim \|\Delta_j \Lambda^\ell (u \cdot \nabla u)\|_{L^r} + \|\Delta_j \Lambda^\ell (\Delta \phi \nabla \phi)\|_{L^r}, \quad (4.18)$$

$$\frac{d}{dt} \|\Delta_j \Lambda^{\ell-1} v\|_{L^r} + \kappa 2^{2j} \|\Delta_j \Lambda^{\ell-1} v\|_{L^r} \lesssim \|\Delta_j \Lambda^{\ell-1} (u \cdot \nabla v)\|_{L^r} + \|\Delta_j \Lambda^{\ell-1} \nabla \cdot (v \nabla \phi)\|_{L^r}. \quad (4.19)$$

Taking l^1 norm to (4.18) and (4.19), respectively, and using Lemma 5.2, we see that

$$\frac{d}{dt} \|u\|_{\dot{B}_{r,1}^\ell} + \kappa \|u\|_{\dot{B}_{r,1}^{\ell+2}} \lesssim \|u \cdot \nabla u\|_{\dot{B}_{r,1}^\ell} + \|\Delta \phi \nabla \phi\|_{\dot{B}_{r,1}^\ell}, \quad (4.20)$$

$$\frac{d}{dt} \|v\|_{\dot{B}_{r,1}^{\ell-1}} + \kappa \|v\|_{\dot{B}_{r,1}^{\ell+1}} \lesssim \|u \cdot \nabla v\|_{\dot{B}_{r,1}^{\ell-1}} + \|\nabla \cdot (v \nabla \phi)\|_{\dot{B}_{r,1}^{\ell-1}}. \quad (4.21)$$

In order to finish the proof of Proposition 4.6, the case $\ell > 0$ is simple because of $\dot{B}_{r,1}^\ell(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a Banach algebra, and $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,1}^0(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for all $1 \leq p \leq \infty$, we obtain by using Lemma 5.2 that

$$\|u \cdot \nabla u\|_{\dot{B}_{r,1}^\ell} \lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{\dot{B}_{r,1}^\ell} \|u\|_{L^\infty}$$

$$\begin{aligned}
&\lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|\nabla u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|u\|_{\dot{B}_{r,1}^{\ell+1}} \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\
&\lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|u\|_{\dot{B}_{r,1}^{\frac{1}{2}}} \|u\|_{\dot{B}_{r,1}^{\ell+2}} \|u\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}^{\frac{1}{2}} \\
&\leq \frac{\kappa}{2} \|u\|_{\dot{B}_{r,1}^{\ell+2}} + C \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|u\|_{\dot{B}_{r,1}^\ell},
\end{aligned}$$

where we have used the interpolation inequalities:

$$\|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{\frac{1}{2}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}^{\frac{1}{2}}, \quad \|u\|_{\dot{B}_{r,1}^{\ell+1}} \lesssim \|u\|_{\dot{B}_{r,1}^{\ell+2}}^{\frac{1}{2}} \|u\|_{\dot{B}_{r,1}^\ell}^{\frac{1}{2}}.$$

Similarly, we have

$$\begin{aligned}
\|\Delta \phi \nabla \phi\|_{\dot{B}_{r,1}^\ell} &= \|(v-w) \nabla (-\Delta)^{-1} (w-v)\|_{\dot{B}_{r,1}^\ell} \\
&\lesssim \|\nabla (-\Delta)^{-1} (w-v)\|_{\dot{B}_{r,1}^\ell} \|v-w\|_{L^\infty} + \|v-w\|_{\dot{B}_{r,1}^\ell} \|\nabla (-\Delta)^{-1} (w-v)\|_{L^\infty} \\
&\lesssim \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}} \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(v,w)\|_{\dot{B}_{r,1}^\ell} \|(v,w)\|_{\dot{B}_{q,1}^{-1+\frac{3}{q}}} \\
&\lesssim \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}} \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}}^{\frac{1}{2}} \|(v,w)\|_{\dot{B}_{r,1}^{\ell+1}}^{\frac{1}{2}} \|(v,w)\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}^{\frac{1}{2}} \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}^{\frac{1}{2}} \\
&\leq \frac{\kappa}{6} \|(v,w)\|_{\dot{B}_{r,1}^{\ell+1}} + C \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}};
\end{aligned}$$

$$\begin{aligned}
\|u \cdot \nabla v\|_{\dot{B}_{r,1}^{\ell-1}} &\approx \|uv\|_{\dot{B}_{r,1}^\ell} \lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|v\|_{\dot{B}_{r,1}^\ell} \\
&\lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{\frac{1}{2}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}^{\frac{1}{2}} \|v\|_{\dot{B}_{r,1}^{\ell-1}}^{\frac{1}{2}} \|v\|_{\dot{B}_{r,1}^{\ell+1}}^{\frac{1}{2}} \\
&\leq \frac{\kappa}{6} \|v\|_{\dot{B}_{r,1}^{\ell+1}} + C \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{r,1}^\ell} + C \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|v\|_{\dot{B}_{r,1}^{\ell-1}};
\end{aligned}$$

$$\begin{aligned}
\|\nabla \cdot (v \nabla \phi)\|_{\dot{B}_{r,1}^{\ell-1}} &\approx \|v \nabla \phi\|_{\dot{B}_{r,1}^\ell} = \|v \nabla (-\Delta)^{-1} (w-v)\|_{\dot{B}_{r,1}^\ell} \\
&\lesssim \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}} \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(v,w)\|_{\dot{B}_{q,1}^{-1+\frac{3}{q}}} \|v\|_{\dot{B}_{r,1}^\ell} \\
&\lesssim \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}} \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(v,w)\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}^{\frac{1}{2}} \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}^{\frac{1}{2}} \|v\|_{\dot{B}_{r,1}^{\ell-1}}^{\frac{1}{2}} \|v\|_{\dot{B}_{r,1}^{\ell+1}}^{\frac{1}{2}} \\
&\lesssim \frac{\kappa}{6} \|v\|_{\dot{B}_{r,1}^{\ell+1}} + C \|(v,w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|(v,w)\|_{\dot{B}_{r,1}^{\ell-1}}.
\end{aligned}$$

On the other hand, in the case $\ell \leq 0$, recall that

$$\frac{3}{p} + \ell > 3 \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad \frac{3}{q} + \ell > 3 \max\{0, \frac{1}{q} + \frac{1}{r} - 1\}.$$

Applying Lemma 5.3 with $f = u$, $g = \nabla u$, $s_1 = \frac{3}{p}$, $s_2 = \ell$, $p_1 = p$, $p_2 = r$, we have

$$\|u \cdot \nabla u\|_{\dot{B}_{r,1}^\ell} \lesssim \|u\|_{\dot{B}_{r,1}^\ell} \|\nabla u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} \|u\|_{\dot{B}_{r,1}^\ell};$$

Applying Lemma 5.3 with $f = v - w$, $g = \nabla(-\Delta)^{-1}(w - v)$, $s_1 = \frac{3}{q}$, $s_2 = \ell$, $p_1 = q$, $p_2 = r$, we have

$$\begin{aligned} \|\Delta\phi\nabla\phi\|_{\dot{B}_{r,1}^\ell} &= \|(v - w)\nabla(-\Delta)^{-1}(w - v)\|_{\dot{B}_{r,1}^\ell} \\ &\lesssim \|\nabla(-\Delta)^{-1}(w - v)\|_{\dot{B}_{r,1}^\ell} \|v - w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \lesssim \|(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}; \end{aligned}$$

Applying Lemma 5.3 with $f = u$, $g = v$, $s_1 = \frac{3}{q}$, $s_2 = \ell$, $p_1 = q$, $p_2 = r$, we have

$$\|u \cdot \nabla v\|_{\dot{B}_{r,1}^{\ell-1}} \approx \|uv\|_{\dot{B}_{r,1}^\ell} \lesssim \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{r,1}^\ell};$$

Applying Lemma 5.3 with $f = \nabla(-\Delta)^{-1}(w - v)$, $g = v$, $s_1 = \frac{3}{q}$, $s_2 = \ell$, $p_1 = q$, $p_2 = r$, we have

$$\|\nabla \cdot (v\nabla\phi)\|_{\dot{B}_{r,1}^{\ell-1}} \approx \|v\nabla\phi\|_{\dot{B}_{r,1}^\ell} \lesssim \|\nabla(-\Delta)^{-1}(w - v)\|_{\dot{B}_{r,1}^\ell} \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \lesssim \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}.$$

Therefore, we conclude that

$$\begin{aligned} \frac{d}{dt} \|u\|_{\dot{B}_{r,1}^\ell} + \frac{\kappa}{2} \|u\|_{\dot{B}_{r,1}^{\ell+2}} &\leq \frac{\kappa}{6} \|(v, w)\|_{\dot{B}_{r,1}^{\ell+1}} \\ &\quad + C(\|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) (\|u\|_{\dot{B}_{r,1}^\ell} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}); \end{aligned} \quad (4.22)$$

$$\frac{d}{dt} \|v\|_{\dot{B}_{r,1}^{\ell-1}} + \frac{2\kappa}{3} \|v\|_{\dot{B}_{r,1}^{\ell+1}} \leq C(\|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) (\|u\|_{\dot{B}_{r,1}^\ell} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}). \quad (4.23)$$

Similarly,

$$\frac{d}{dt} \|w\|_{\dot{B}_{r,1}^{\ell-1}} + \frac{2\kappa}{3} \|w\|_{\dot{B}_{r,1}^{\ell+1}} \leq C(\|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) (\|u\|_{\dot{B}_{r,1}^\ell} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}). \quad (4.24)$$

By adding (4.22)–(4.24) together, we finally obtain that

$$\begin{aligned} \frac{d}{dt} (\|u\|_{\dot{B}_{r,1}^\ell} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}) &+ \frac{\kappa}{2} (\|u\|_{\dot{B}_{r,1}^{\ell+2}} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell+1}}) \\ &\leq C(\|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} + \|(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) (\|u\|_{\dot{B}_{r,1}^\ell} + \|(v, w)\|_{\dot{B}_{r,1}^{\ell-1}}). \end{aligned}$$

This yields (4.17) immediately. We complete the proof of Proposition 4.6. \square

4.3 Proof of Theorem 3.1

Now we present the proof of Theorem 3.1. We first mention that Proposition 4.6 implies (3.4) directly, so it suffices to prove (3.5). For this purpose, for any $s > 0$ such that

$$\frac{3}{p} - s > 3 \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad \frac{3}{q} - s > 3 \max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

by choosing $\ell = -s$ in Proposition 4.6, we see that for all $t \geq 0$,

$$\|u(t)\|_{\dot{B}_{r,1}^{-s}} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{-s-1}} \leq C \left(\|u_0\|_{\dot{B}_{r,1}^{-s}} + \|(v_0, w_0)\|_{\dot{B}_{r,1}^{-s-1}} \right) \leq C_0. \quad (4.25)$$

This particularly gives (3.5) with $\ell = -s$. On the other hand, for any $\ell \in (-s, N]$, by interpolation inequalities in Lemma 5.2, we have for all $t \geq 0$,

$$\begin{aligned}\|u(t)\|_{\dot{B}_{r,1}^\ell} &\leq C\|u(t)\|_{\dot{B}_{r,1}^{-s}}^{\frac{2}{\ell+s+2}}\|u(t)\|_{\dot{B}_{r,1}^{\ell+2}}^{1-\frac{2}{\ell+s+2}}, \\ \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}} &\leq C\|(v(t), w(t))\|_{\dot{B}_{r,1}^{-s-1}}^{\frac{2}{\ell+s+2}}\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell+1}}^{1-\frac{2}{\ell+s+2}}.\end{aligned}$$

This together with (4.25) implies that

$$\begin{aligned}\|u(t)\|_{\dot{B}_{r,1}^{\ell+2}} &\geq C\|u(t)\|_{\dot{B}_{r,1}^\ell}^{1+\frac{2}{\ell+s}}\|u(t)\|_{\dot{B}_{r,1}^{-s}}^{-\frac{2}{\ell+s}} \geq C\|u(t)\|_{\dot{B}_{r,1}^\ell}^{1+\frac{2}{\ell+s}}, \\ \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell+1}} &\geq C\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}}^{1+\frac{2}{\ell+s}}\|(v(t), w(t))\|_{\dot{B}_{r,1}^{-s-1}}^{-\frac{2}{\ell+s}} \geq C\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}}^{1+\frac{2}{\ell+s}}.\end{aligned}$$

It follows that

$$\begin{aligned}\|u(t)\|_{\dot{B}_{r,1}^{\ell+2}} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell+1}} &\geq C(\|u(t)\|_{\dot{B}_{r,1}^\ell} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}})^{1+\frac{2}{\ell+s}} \\ &= C\mathcal{F}(t)^{1+\frac{2}{\ell+s}}.\end{aligned}\tag{4.26}$$

Plugging (4.26) into (4.17), we see that

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{F}(t)) + Ce^{-KY(t)}\mathcal{F}(t)^{1+\frac{2}{\ell+s}} \leq 0,$$

which combining the fact that the function $Y(t)$ is positive along time evolution yields that

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{F}(t)) + C(e^{-KY(t)}\mathcal{F}(t))^{1+\frac{2}{\ell+s}} \leq 0.\tag{4.27}$$

Solving this differential inequality directly, we obtain

$$\mathcal{F}(t) \leq e^{KY(t)} \left(\mathcal{F}(0)^{-\frac{2}{\ell+s}} + \frac{2Ct}{\ell+s} \right)^{-\frac{\ell+s}{2}}.$$

Note that the function $Y(t)$ is bounded by the initial data in Proposition 4.1. Hence, we see that for all $t \geq 0$, there exists a constant C_0 such that

$$\|u(t)\|_{\dot{B}_{r,1}^\ell} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}} \leq C_0 (1+t)^{-\frac{\ell+s}{2}}.\tag{4.28}$$

We complete the proof of Theorem 3.1, as desired.

4.4 Proof of Theorem 3.2

Since $1 < r \leq \min\{p, q\}$, we infer from the imbedding results in Lemma 5.2 that

$$\dot{B}_{r,1}^{-s}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,1}^{-s-3(\frac{1}{r}-\frac{1}{p})}(\mathbb{R}^3) \quad \text{and} \quad \dot{B}_{r,1}^{-s-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q,1}^{-s-1-3(\frac{1}{r}-\frac{1}{q})}(\mathbb{R}^3),$$

which together with (4.25) leads to for all $t \geq 0$,

$$\|u(t)\|_{\dot{B}_{r,1}^{-s-3(\frac{1}{r}-\frac{1}{p})}} + \|(v(t), w(t))\|_{\dot{B}_{r,1}^{-s-1-3(\frac{1}{r}-\frac{1}{q})}} \leq C_0.\tag{4.29}$$

On the other hand, for any $s \geq \max\{0, 2 - \frac{3}{r}\}$, by interpolation inequalities in Lemma 5.2, we have for all $t \geq 0$,

$$\begin{aligned} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} &\leq C \|u(t)\|_{\dot{B}_{p,1}^{-s-3(\frac{1}{r}-\frac{1}{p})}}^{\frac{2}{s+\frac{3}{r}+1}} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1-\frac{2}{s+\frac{3}{r}+1}}, \\ \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} &\leq C \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-s-1-3(\frac{1}{r}-\frac{1}{q})}}^{\frac{2}{s+\frac{3}{r}+1}} \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-\frac{3}{q}}}^{1-\frac{2}{s+\frac{3}{r}+1}}. \end{aligned}$$

This together with (4.29) implies that

$$\begin{aligned} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} &\geq C \|u(t)\|_{\dot{B}_{p,1}^{-s-3(\frac{1}{r}-\frac{1}{p})}}^{\frac{2}{s+\frac{3}{r}-1}} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1+\frac{2}{s+\frac{3}{r}-1}} \geq C \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1+\frac{2}{s+\frac{3}{r}-1}}, \\ \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-\frac{3}{q}}} &\geq C \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-s-1-3(\frac{1}{r}-\frac{1}{q})}}^{\frac{2}{s+\frac{3}{r}-1}} \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}^{1+\frac{2}{s+\frac{3}{r}-1}} \geq C \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}^{1+\frac{2}{s+\frac{3}{r}-1}}. \end{aligned}$$

It follows that

$$\begin{aligned} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-\frac{3}{q}}} &\geq C (\|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}})^{1+\frac{2}{s+\frac{3}{r}-1}} \\ &= C \mathcal{E}(t)^{1+\frac{2}{s+\frac{3}{r}-1}}. \end{aligned} \quad (4.30)$$

Plugging (4.30) into (4.1), by using the function $Y(t)$ is positive along time evolution, we obtain

$$\frac{d}{dt} (e^{-KY(t)} \mathcal{E}(t)) + C(e^{-KY(t)} \mathcal{E}(t))^{1+\frac{2}{s+\frac{3}{r}-1}} \leq 0. \quad (4.31)$$

Solving this differential inequality directly, we obtain

$$\mathcal{E}(t) \leq e^{KY(t)} \left(\mathcal{E}(0)^{-\frac{2}{s+\frac{3}{r}-1}} + \frac{2Ct}{s+\frac{3}{r}-1} \right)^{-\frac{s+\frac{3}{r}-1}{2}}.$$

Since $Y(t)$ is bounded by the initial data in Proposition 4.1, there exists a constant C_0 such that for all $t \geq 0$,

$$\|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}} \leq C_0 (1+t)^{-\frac{s+\frac{3}{r}-1}{2}}. \quad (4.32)$$

Notice that (4.32) gives in particular (3.6) with $\ell = -1 + \frac{3}{p}$, and (3.7) with $\ell - 1 = -2 + \frac{3}{q}$, respectively. Finally, for any $\ell \in [-s - 3(\frac{1}{r} - \frac{1}{p}), -1 + \frac{3}{p}]$, by using interpolation inequality in Lemma 5.2, we see that

$$\|u(t)\|_{\dot{B}_{r,1}^{\ell}} \leq C \|u(t)\|_{\dot{B}_{p,1}^{-s-3(\frac{1}{r}-\frac{1}{p})}}^{\frac{\frac{3}{p}-1-\ell}{s+\frac{3}{r}-1}} \|u(t)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{\frac{\ell+s+3(\frac{1}{r}-\frac{1}{p})}{s+\frac{3}{r}-1}},$$

which combining (4.29) and (4.32) implies that

$$\|u(t)\|_{\dot{B}_{r,1}^{\ell}} \leq C_0 (1+t)^{-\left(\frac{\ell+s}{2}\right) - \frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}.$$

Similarly, for any $\ell \in [-s-1-3(\frac{1}{r}-\frac{1}{q}), -2+\frac{3}{q}]$, there exists a constant C_0 such that for all $t \geq 0$,

$$\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}} \leq C \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-s-1-3(\frac{1}{r}-\frac{1}{q})}}^{\frac{\frac{3}{q}-1-\ell}{s+\frac{3}{r}-1}} \|(v(t), w(t))\|_{\dot{B}_{q,1}^{-2+\frac{3}{q}}}^{\frac{\ell+s+3(\frac{1}{r}-\frac{1}{q})}{s+\frac{3}{r}-1}},$$

which combining (4.29) and (4.32) again leads to

$$\|(v(t), w(t))\|_{\dot{B}_{r,1}^{\ell-1}} \leq C_0(1+t)^{-(\frac{\ell+s}{2})-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}.$$

We complete the proof of Theorem 3.2, as desired.

5 Appendix

We first recall some crucial analytic tools used in the proofs of Theorems 3.1 and 3.2, then give a sketched proof for global existence part in Theorem 3.1.

5.1 Useful lemmas

Lemma 5.1 ([1], [6]) *Let \mathcal{B} be a ball, and \mathcal{C} a ring in \mathbb{R}^3 . There exists a constant C such that for any positive real number λ , any nonnegative integer k and any couple of real numbers (a, b) with $1 \leq a \leq b \leq \infty$, we have*

$$\text{supp } \hat{f} \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+3(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a}, \quad (5.1)$$

$$\text{supp } \hat{f} \subset \lambda \mathcal{C} \Rightarrow C^{-1-k} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{1+k} \lambda^k \|f\|_{L^a}. \quad (5.2)$$

Let us now state some basic properties of Besov spaces (see [1], [6]).

Lemma 5.2 ([1], [6]) *The following properties hold:*

i) *Density: The set $C_0^\infty(\mathbb{R}^3)$ is dense in $\dot{B}_{p,r}^s(\mathbb{R}^3)$ if $|s| < \frac{3}{p}$ and $1 \leq p, r < \infty$ or $s = \frac{3}{p}$ and $r = 1$.*

ii) *Derivatives: There exists a universal constant C such that*

$$C^{-1} \|u\|_{\dot{B}_{p,r}^s} \leq \|\nabla u\|_{\dot{B}_{p,r}^{s-1}} \leq C \|u\|_{\dot{B}_{p,r}^s}.$$

iii) *Fractional derivative: Let $\Lambda = \sqrt{-\Delta}$ and $\sigma \in \mathbb{R}$. Then the operator Λ^σ is an isomorphism from $\dot{B}_{p,r}^s(\mathbb{R}^3)$ to $\dot{B}_{p,r}^{s-\sigma}(\mathbb{R}^3)$.*

iv) *Algebraic properties: For $s > 0$, $\dot{B}_{p,r}^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is an algebra. Moreover, $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,1}^0(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, and for any $f, g \in \dot{B}_{p,r}^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we have*

$$\|fg\|_{\dot{B}_{p,r}^s} \leq \|f\|_{\dot{B}_{p,r}^s} \|g\|_{L^\infty} + \|g\|_{\dot{B}_{p,r}^s} \|f\|_{L^\infty}.$$

- v) *Imbedding:* For $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, we have the continuous imbedding $\dot{B}_{p_1, r_1}^s(\mathbb{R}^3) \hookrightarrow \dot{B}_{p_2, r_2}^{s-3(\frac{1}{p_1}-\frac{1}{p_2})}(\mathbb{R}^3)$.
- vi) *Interpolation:* For $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$ and $\theta \in (0, 1)$, there exists a constant C such that

$$\|u\|_{\dot{B}_{p, r}^{s_1\theta+s_2(1-\theta)}} \leq C \|u\|_{\dot{B}_{p, r}^{s_1}}^\theta \|u\|_{\dot{B}_{p, r}^{s_2}}^{1-\theta}.$$

Lemma 5.3 *Let $1 \leq p_1, p_2 \leq \infty$, and $s_1 \leq \frac{3}{p_1}$, $s_2 \leq \min\{\frac{3}{p_1}, \frac{3}{p_2}\}$ with $s_1 + s_2 > 3\max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1)$. Assume that $f \in \dot{B}_{p_1, 1}^{s_1}(\mathbb{R}^3)$, $g \in \dot{B}_{p_2, 1}^{s_2}(\mathbb{R}^3)$. Then $fg \in \dot{B}_{p_2, 1}^{s_1+s_2-\frac{3}{p_1}}(\mathbb{R}^3)$, and there exists a positive constant C such that*

$$\|fg\|_{\dot{B}_{p_2, 1}^{s_1+s_2-\frac{3}{p_1}}} \leq C \|f\|_{\dot{B}_{p_1, 1}^{s_1}} \|g\|_{\dot{B}_{p_2, 1}^{s_2}}. \quad (5.3)$$

Proof. The ideas comes essentially from [6]. Thanks to Bony's paraproduct decomposition, we have

$$fg = T_f g + T_g f + R(f, g).$$

Applying Lemma 5.1 gives

$$\begin{aligned} \|\Delta_j T_f g\|_{L^{p_2}} &\lesssim \sum_{|j'-j| \leq 4} \|S_{j'-1} f\|_{L^\infty} \|\Delta_{j'} g\|_{L^{p_2}} \\ &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{(\frac{3}{p_1}-s_1)k} 2^{s_1 k} \|\Delta_k f\|_{L^{p_1}} \|\Delta_{j'} g\|_{L^{p_2}} \\ &\lesssim 2^{(\frac{3}{p_1}-s_1-s_2)j} d_j \|f\|_{\dot{B}_{p_1, 1}^{s_1}} \|g\|_{\dot{B}_{p_2, 1}^{s_2}}. \end{aligned} \quad (5.4)$$

For the term $T_g f$, in the case that $1 \leq p_1 \leq p_2$, it follows from Lemma 5.1 that

$$\begin{aligned} \|\Delta_j T_g f\|_{L^{p_2}} &\lesssim \sum_{|j'-j| \leq 4} 2^{3(\frac{1}{p_1}-\frac{1}{p_2})j'} \|S_{j'-1} g\|_{L^\infty} \|\Delta_{j'} f\|_{L^{p_1}} \\ &\lesssim \sum_{|j'-j| \leq 4} 2^{3(\frac{1}{p_1}-\frac{1}{p_2})j'} \sum_{k \leq j'-2} 2^{(\frac{3}{p_2}-s_2)k} 2^{s_2 k} \|\Delta_k g\|_{L^{p_2}} \|\Delta_{j'} f\|_{L^{p_1}} \\ &\lesssim 2^{(\frac{3}{p_1}-s_1-s_2)j} d_j \|f\|_{\dot{B}_{p_1, 1}^{s_1}} \|g\|_{\dot{B}_{p_2, 1}^{s_2}}, \end{aligned} \quad (5.5)$$

while in the case that $p_2 < p_1$, we have

$$\begin{aligned} \|\Delta_j T_g f\|_{L^{p_2}} &\lesssim \sum_{|j'-j| \leq 4} \|S_{j'-1} g\|_{L^{\frac{p_1 p_2}{p_1-p_2}}} \|\Delta_{j'} f\|_{L^{p_1}} \\ &\lesssim \sum_{|j'-j| \leq 4} \sum_{k \leq j'-2} 2^{(\frac{3}{p_1}-s_2)k} 2^{s_2 k} \|\Delta_k g\|_{L^{p_2}} \|\Delta_{j'} f\|_{L^{p_1}} \\ &\lesssim 2^{(\frac{3}{p_1}-s_1-s_2)j} d_j \|f\|_{\dot{B}_{p_1, 1}^{s_1}} \|g\|_{\dot{B}_{p_2, 1}^{s_2}}. \end{aligned} \quad (5.6)$$

Here estimates (5.4), (5.5) and (5.6) are verified since $s_1 \leq \frac{3}{p_1}$, $s_2 \leq \min\{\frac{3}{p_1}, \frac{3}{p_2}\}$. Finally, in the case that $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, using Lemma 5.1 again yields that

$$\|\Delta_j R(f, g)\|_{L^{p_2}} \lesssim 2^{\frac{3j}{p_1}} \sum_{j' \geq j-N_0} \|\Delta_{j'} f\|_{L^{p_1}} \|\tilde{\Delta}_{j'} g\|_{L^{p_2}}$$

$$\begin{aligned}
&\lesssim 2^{\frac{3j}{p_1}} \sum_{j' \geq j - N_0} 2^{-(s_1+s_2)j'} 2^{s_1 j'} \|\Delta_{j'} f\|_{L^{p_1}} 2^{s_2 j'} \|\tilde{\Delta}_{j'} g\|_{L^{p_2}} \\
&\lesssim 2^{(\frac{3}{p_1} - s_1 - s_2)j} d_j \|f\|_{\dot{B}_{p_1,1}^{s_1}} \|g\|_{\dot{B}_{p_2,1}^{s_2}}.
\end{aligned} \tag{5.7}$$

If $\frac{1}{p_1} + \frac{1}{p_2} > 1$, then we find p'_2 such that $\frac{1}{p_2} + \frac{1}{p'_2} = 1$, and

$$\begin{aligned}
\|\Delta_j R(f, g)\|_{L^{p_2}} &\lesssim 2^{3(1-\frac{1}{p_2})j} \sum_{j' \geq j - N_0} \|\Delta_{j'} f \tilde{\Delta}_{j'} g\|_{L^1} \\
&\lesssim 2^{3(1-\frac{1}{p_2})j} \sum_{j' \geq j - N_0} \|\Delta_{j'} f\|_{L^{p'_2}} \|\tilde{\Delta}_{j'} g\|_{L^{p_2}} \\
&\lesssim 2^{3(1-\frac{1}{p_2})j} \sum_{j' \geq j - N_0} 2^{3(\frac{1}{p_1} + \frac{1}{p_2} - 1)j'} \|\Delta_{j'} f\|_{L^{p_1}} \|\tilde{\Delta}_{j'} g\|_{L^{p_2}} \\
&\lesssim 2^{3(1-\frac{1}{p_2})j} \sum_{j' \geq j - N_0} 2^{3(\frac{1}{p_1} + \frac{1}{p_2} - 1)j'} 2^{-(s_1+s_2)j'} 2^{s_1 j'} \|\Delta_{j'} f\|_{L^{p_1}} 2^{s_2 j'} \|\tilde{\Delta}_{j'} g\|_{L^{p_2}} \\
&\lesssim 2^{(\frac{3}{p_1} - s_1 - s_2)j} d_j \|f\|_{\dot{B}_{p_1,1}^{s_1}} \|g\|_{\dot{B}_{p_2,1}^{s_2}}.
\end{aligned} \tag{5.8}$$

Here estimates (5.7) and (5.8) are verified since $s_1 + s_2 > 3 \max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1)$. We complete the proof of Lemma 5.3. \square

5.2 Global existence with small initial data

In this section, we sketch the proof of global existence part in Theorem 3.1. The approach is similar to that of [24]. The only difficulty lies in estimations of the nonlinear terms $\Delta\phi\nabla\phi$ and $u \cdot \nabla v$, which if $1 \leq q \leq p$, we have proved the desired bilinear estimates in [24]. In the case of $p < q$, the proof comes essentially from the approach used in Lemma 4.2 to estimate $\Delta\phi\nabla\phi$, and Lemma 4.3 to estimate $u \cdot \nabla v$, where we need only to deal with the time variable by the general principle that the time exponent behaves according to the Hölder inequality, and an additional condition $\frac{1}{q} - \frac{1}{p} > -\min\{\frac{1}{3}, \frac{1}{2p}\}$ is needed. More precisely, since

$$\Delta\phi\nabla\phi = -(v-w)\nabla(-\Delta)^{-1}(v-w),$$

we can show that

$$\|v\nabla(-\Delta)^{-1}w + w\nabla(-\Delta)^{-1}v\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \lesssim \|v\|_{\mathcal{L}_t^\infty(\dot{B}_{q,1}^{-2+\frac{3}{q}})} \|w\|_{L_t^1(\dot{B}_{q,1}^{\frac{3}{q}})} + \|w\|_{\mathcal{L}_t^\infty(\dot{B}_{q,1}^{-2+\frac{3}{q}})} \|v\|_{L_t^1(\dot{B}_{q,1}^{\frac{3}{q}})}$$

and

$$\|u \cdot \nabla v\|_{L_t^1(\dot{B}_{q,1}^{-2+\frac{3}{q}})} \lesssim \|u\|_{\mathcal{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|v\|_{L_t^1(\dot{B}_{q,1}^{\frac{3}{q}})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \|v\|_{\mathcal{L}_t^\infty(\dot{B}_{q,1}^{-2+\frac{3}{q}})}.$$

Based on these two desired bilinear estimates, we can follow the approach used in [24] to prove that if $\|(u_0, v_0, w_0)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}} \times (\dot{B}_{q,1}^{-2+\frac{3}{q}})^2}$ is sufficiently small, then the system (3.1)–(3.2) admits a unique global solution. We complete the proof, as desired.

Acknowledgments. J. Zhao is partially supported by the National Natural Science Foundation of China (11371294), the Fundamental Research Funds for the Central Universities (2014YB031) and

the Fundamental Research Project of Natural Science in Shaanxi Province–Young Talent Project (2015JQ1004). Q. Liu is partially supported by the National Natural Science Foundation of China (11326155, 11401202), the Scientific Research Fund of Hunan Provincial Education Department (14B117) and the China Postdoctoral Science Foundation (2015M570053).

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